

MULTI-VALUED LOGIC GATES, CONTINUOUS SENSITIVITY, REVERSIBILITY, AND THRESHOLD FUNCTIONS

ASLI GÜÇLÜKAN İLHAN AND ÖZGÜN ÜNLÜ

ABSTRACT. We define an invariant of a multi-valued logic gate by considering the number of certain threshold functions associated with the gate. We call this invariant the continuous sensitivity of the gate. We discuss a method for analysing continuous sensitivity of a multi-valued logic gate by using experimental data about the gate. In particular, we will show that this invariant provides a lower bound for the sensitivity of a boolean function considered as a multi-valued logic gate. We also discuss how continuous sensitivity can be used to understand the reversibility of the Fourier series expansion of a multi-valued logic gate.

CONTENTS

1. Introduction	1
2. Definition of Continuous Sensitivity	3
3. Reversibility	6
4. Computing Continuous Sensitivity	9
References	14

1. INTRODUCTION

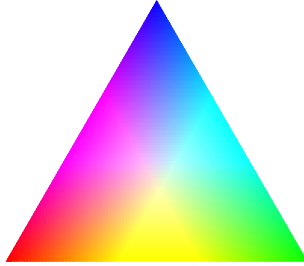
An n -valued logic is a propositional calculus in which the set of truth values has n elements see [5], [19], [22], [23], [24]. In particular, classical logic can be considered as a 2-valued logic in which the truth values are “true” and “false”. Most of the computers have gates which can perform classical logic operations like conjunction (AND), disjunction (OR), negation (NOT). There are increasingly more electronic devices which uses n -valued logic. For example, there are computers like SETUN which uses 3-valued logic and there are memory technologies like Intel StrataFlash which uses multiple levels per cell to allow more bits to be stored using the same number of transistors. However, a better mathematical model to explain the logical operations we see in nature would be multi-valued logic. Multi-valued logic is a propositional calculus where the logical operations have input variables and an output variable with possibly different sets of truth values. For example, while buying a used car our brain takes inputs with many possible values like the color of the car (“red”, “green”, “blue”, ...), the price of the car (“cheap”, “expensive”, ...), the number of

dealers visited (“1”, “2”, “3”, ...) and the logical conclusion is many valued as well (“Yes! I will buy it.”, “No! I will not buy it.”, “I might buy it later.”, ...). For some applications of multi-valued logic, see studies done by the MVSIS group at Berkeley.

In this paper we will consider truth values as vectors in finite dimensional vector spaces. In nature, logic gates communicate with each other using signals and most of the time these signals are analog signals. We know that we can model an analog signal as a point in a Hilbert Space. However, when a natural observer receives signals and tries to make measurements about them a certain set of signals plays a key role. For example, operators correspond to observable quantities in quantum mechanics and the only possible result of the measurements are the eigenvalues of the eigenvectors of the operator. Here we will consider cases where this certain set of signals is finite. Depending on the nature of the observer amplitudes of the signals and/or accumulated results of the measurements over a certain block of time might play a role in final processing in the gate hence we will consider the convex hulls created by these finite sets of signals in the corresponding Hilbert spaces. For example, our eyes can only observe red, green, blue light and lack of light namely black. Hence the convex hull created by these can be considered as a 3-simplex Δ^3 where in general we have

$$\Delta^n = \{(t_0, t_1, \dots, t_n) \mid \sum_{i=0}^n t_i = 1\}$$

Here, if “black” corresponds to the point $(1, 0, 0, 0)$. Then for $t_0 = 0$ we obtain a color triangle which could be considered as Δ^2 .



One could take the truth values for color as the vertices in a barycentric subdivision of the triangle above in particular if “red” corresponds to $(1, 0, 0)$ and “blue” corresponds to $(0, 0, 1)$ then purple will correspond to $(1/2, 0, 1/2)$.

On these convex hulls we will be considering Fourier series expansions of multi-valued logic gates. For the Fourier series expansions of Boolean functions see [6], [7], [10], [15], [17], [18], [20]. The input will come from a product of simplices as we can consider each convex hull as a union of simplices. More precisely a multi-valued logic gate ϕ is a function from a product of sets $T_1 \times T_2 \times \dots \times T_k$ to a set $T = \{v_1, v_2, \dots, v_n\}$ of n vectors in \mathbb{R}^m where the set T_i is the set of possible truth values of the i^{th} input variable and the set T as the set of possible truth values of the output. We define

the Fourier-expansion of the multi-valued logic gate as the multilinear function

$$\bar{\phi} : \Delta^{n_1-1} \times \Delta^{n_2-1} \times \dots \times \Delta^{n_k-1} \rightarrow \mathbb{R}^m$$

given by

$$\bar{\phi}((t_{1,j})_{j \in T_1}, (t_{2,j})_{j \in T_2}, \dots, (t_{k,j})_{j \in T_k}) = \sum_{j \in T} \left(\sum_{\phi(j_1, \dots, j_k) = v_j} \prod_{s=1}^k t_{s, j_s} \right) v_j$$

where n_i is the number of elements in T_i . Notice that the output of this gate lives in the convex hull spanned by the vectors $T = \{v_1, \dots, v_n\}$.

One can wonder the sensitivity of a multi-valued logic gate to change in its input variables. For example DNA codes can be considered as an input to a 4-valued logic gate and one can ask on which parts of the inputs does our eye color depend. In this paper we will define continuous sensitivity of a multi-valued logic gate as cardinalities of certain sets of threshold functions and discuss a method to analyse continuous sensitivity of a multi-valued logic gate using experimental data which means a list of input and output data for the multi-valued logic gate. As an application we will use such an argument to show that continuous sensitivity provides a lower bound for sensitivity of a boolean function (see [2], [3], [9], [11], [12], [16], [28], [29], [33], [35]) considered as a multi-valued logic gate. For more advanced applications, one of our main results will be Proposition 4.7 which is about cardinalities of certain threshold functions.

Another important issue to consider about multi-valued logic gates is reversibility (see [1], [4], [8], [13], [14], [21], [25], [26], [27], [30], [31], [32], [34], [36]). For example, one can use reversible gates to build energy efficient computer or for a television producer it is important to be able go back and forth between different mathematical models of colors that a human can see. To show that a gate is reversible one has to show that the gate is one-to-one as a function. Our second main result in this paper is Theorem 3.1. This result can be used to show that a multi-valued logic gate is one-to-one when continuous sensitivity is equal to a certain number (see Corollary 3.2). As an example, we will discuss a logical gate that sends $\Delta^1 \times \Delta^1 \times \Delta^1$ to Δ^3 and can be considered as a gate which translates a digital signal about color to the color tetrahedron mentioned above (see Example 3.8).

2. DEFINITION OF CONTINUOUS SENSITIVITY

In this section we define continuous sensitivity of a multi-valued logic gate. For x in \mathbb{R} , we have

$$\text{sgn}(x) = \begin{cases} +1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0; \end{cases}$$

Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function and C be a nonempty subset of \mathbb{R}^n . The composition $\text{sgn} \circ g$ is a threshold function and in this paper we discuss such functions using the following definition: We define the sign of g over C as follows:

$$\text{sign}_C(g) = \begin{cases} +1, & \text{if } \text{sgn}(g(C)) = \{+1\}; \\ 0, & \text{if } \text{sgn}(g(C)) = \{0\}; \\ -1, & \text{if } \text{sgn}(g(C)) = \{-1\}; \\ u, & \text{otherwise.} \end{cases}$$

If g and C are as above and the function $g(x_1, x_2, \dots, x_n)$ is differentiable on C , then we define the total sign of g over C as follows:

$$\text{Sign}_C(g) = \left\langle \text{sign}_C \left(\frac{\partial g}{\partial x_1} \right), \text{sign}_C \left(\frac{\partial g}{\partial x_2} \right), \dots, \text{sign}_C \left(\frac{\partial g}{\partial x_n} \right) \right\rangle.$$

Let \mathcal{S}_n be the set of all non-zero n -tuples $\langle s_1, s_2, \dots, s_n \rangle$ in $\{-1, 0, 1\}^n$ whose first non-zero term is 1. We say that a tuple $t = \langle t_1, t_2, \dots, t_n \rangle$ in $\{1, 0, -1, u\}^n$ eliminates a tuple $s = \langle s_1, s_2, \dots, s_n \rangle$ in \mathcal{S}_n if the following conditions hold

- i) $t_i \neq 0$ and $s_i \neq 0$ for some i ,
- ii) there exists $k \in \{+1, -1\}$ such that $t_i = ks_i$, for all i with $s_i \neq 0$ and $t_i \neq 0$,
- iii) $s_i = 0$ when $t_i = u$.

For $X \subseteq \mathcal{S}_n$, we define a subset of \mathcal{S}_n as follows:

$$\mathcal{El}(X) = \text{the set of elements of } \mathcal{S}_n \text{ eliminated by an element of } X.$$

Let C be a convex subset of \mathbb{R}^n , M be a smooth manifold, and $f : \mathbb{R}^n \rightarrow M$ be a differentiable function. Now we define a set associated to f as follows:

$$\text{Sens}_C(f) = \left\{ v \in \mathcal{S}_n \mid \begin{array}{l} \text{There exists } \pi : M \rightarrow \mathbb{R} \text{ a differentiable function} \\ \text{such that } \text{Sign}_C(\pi \circ f) \text{ eliminates } v \end{array} \right\}.$$

In other words

$$\text{Sens}_C(f) = \mathcal{El} \{ \text{Sign}_C(\pi \circ f) \mid \pi : M \rightarrow \mathbb{R} \text{ is a differentiable function} \}.$$

Note that the larger the set $\text{Sens}_C(f)$ is the more sensitive the function f is to its input variables. For multi-valued logic gates, we measure the largeness of this set by a number.

Let k be a natural number. Now we will define the Fourier series expansion of a multi-valued logic gate which takes k many inputs. For i in $\{1, 2, \dots, k\}$, let n_i be a natural number and

$$T_i = \{w(i, 0), w(i, 1), \dots, w(i, n_i - 1)\}$$

be the set of possible truth values that we could put in for the i^{th} variable. Let

$$T = \{v_1, v_2, \dots, v_n\}$$

be a set of vectors in \mathbb{R}^m . We will consider the elements in T as truth values of the output. A multi-valued logic gate ϕ is a function from $T_1 \times T_2 \times \dots \times T_k$ to T . Let

$\phi : T_1 \times T_2 \times \cdots \times T_k \rightarrow T$ be such a function. Then the Fourier series expansion of this multi-valued logic gate is the following function

$$\bar{\phi} : \Delta^{n_1-1} \times \Delta^{n_2-1} \times \cdots \times \Delta^{n_k-1} \rightarrow \mathbb{R}^m.$$

given by

$$\bar{\phi}((t_{1,j})_{j=0}^{n_1-1}, (t_{2,j})_{j=0}^{n_2-1}, \dots, (t_{k,j})_{j=0}^{n_k-1}) = \sum_{j=0}^n \left(\sum_{\phi(w(1,j_1), \dots, w(k,j_k))=v_j} \prod_{s=1}^k t_{s,j_s} \right) v_j$$

We define a number $N(\bar{\phi})$ as follows:

$$N(\bar{\phi}) = n_1 + n_2 + \cdots + n_k - k.$$

Take an element z in $T_1 \times T_2 \times \cdots \times T_k$ we can write z in the following form

$$z = (w(1, j(z, 1)), \dots, w(k, j(z, k)))$$

where $0 \leq j(z, i) \leq n_i$. We define $C(\bar{\phi}, z)$ a convex subset of \mathbb{R}^N as follows:

$$C(\bar{\phi}, z) = \Delta_{j(z,1)}^{n_1-1} \times \Delta_{j(z,2)}^{n_2-1} \times \cdots \times \Delta_{j(z,k)}^{n_k-1}$$

where $\Delta_j^m = \{(t_0, \dots, \widehat{t_j}, \dots, t_m) \mid (t_0, t_1, \dots, t_m) \in \Delta^m \text{ and } t_j \neq 0\}$ for natural numbers $j \leq m$. We define a differentiable function $f(\bar{\phi})$ from $\mathbb{R}^{N(\bar{\phi})}$ to \mathbb{R}^m as follows:

$$f(\bar{\phi})(t_{1,1}, \dots, \widehat{t_{1,j(z,1)}}, \dots, \widehat{t_{k,j(z,k)}}, \dots, t_{k,n_k-1}) = \bar{\phi}((t_{1,j})_{j=0}^{n_1-1}, (t_{2,j})_{j=0}^{n_2-1}, \dots, (t_{k,j})_{j=0}^{n_k-1})$$

where the right-hand side is considered to be defined everywhere by seeing each component of the right-hand side as a multilinear polynomial and taking

$$t_{i,j(z,i)} = 1 - \sum_{\substack{1 \leq j \leq n_i-1 \\ j \neq j(z,i)}} t_{i,j}.$$

Now we define the continuous sensitivity of $\bar{\phi}$ at z as follows:

$$\text{cs}(\bar{\phi}, z) = \log_3 \left(3^{N(\bar{\phi})} - 2 \left| \mathcal{E}l(\mathcal{S}_{N(\bar{\phi})} - \text{Sens}_{C(\bar{\phi},z)}(f(\bar{\phi}))) \right| \right).$$

Finally we define continuous sensitivity of $\bar{\phi}$ as follows:

$$\text{cs}(\bar{\phi}) = \max\{\text{cs}(\bar{\phi}, z) \mid z \in T_1 \times T_2 \times \cdots \times T_k\}.$$

The larger this number is the more sensitive the multi-valued logic gate is to its input variables. We will explain this in the next sections.

3. REVERSIBILITY

The main theorem of this section is the following theorem.

Theorem 3.1. *Let C be a convex subset of \mathbb{R}^n , M be a smooth manifold, and $f : \mathbb{R}^n \rightarrow M$ be a differentiable function. If $\text{Sens}_C(f) = \mathcal{S}_n$ then f is one-to-one on C .*

Proof. Assume that there are two distinct points x, y in C such that $f(x) = f(y)$. Let $v = y - x = \langle v_1, \dots, v_n \rangle$. Since $x \neq y$ there exists i such that $v_i \neq 0$. Hence $\text{sgn}(v) = (\text{sgn}(v_1), \dots, \text{sgn}(v_n))$ is in \mathcal{S}_n^* . Therefore there exists a differentiable function $\pi : M \rightarrow \mathbb{R}$ such that $\text{Sign}_C(\pi \circ f)$ eliminates $\text{sgn}(v)$. Let $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ be the linear parametrization of line segment from x to y . Then we get a contradiction as follows:

$$\begin{aligned} 0 &= (\pi \circ f)(y) - (\pi \circ f)(x) = \int_0^1 \frac{d}{dt} ((\pi \circ f \circ \gamma)(t)) dt = \\ &= \int_0^1 \sum_{j=1}^n v_j \left(\frac{\partial}{\partial x_j} (\pi \circ f)|_{\gamma(t)} \right) dt > 0 \end{aligned}$$

because for all j we have

$$v_j \left(\frac{\partial}{\partial x_j} (\pi \circ f)|_{\gamma(t)} \right) \geq 0$$

and equality doesn't hold for at least one j . \square

As consequence of this result we obtain the following result.

Corollary 3.2. *Let $\bar{\phi}$ be the Fourier series expansion of a multi-valued logic gate. If $\text{cs}(\bar{\phi}) = N(\bar{\phi})$ then $\bar{\phi}$ is a one-to-one function on the interior of $\Delta^{n_1-1} \times \Delta^{n_2-1} \times \dots \times \Delta^{n_k-1}$.*

Proof. First notice that $\text{cs}(\bar{\phi}) = N(\bar{\phi})$ means

$$\log_3 \left(3^{N(\bar{\phi})} - 2 \left| \mathcal{E}l(\mathcal{S}_{N(\bar{\phi})} - \text{Sens}_{C(\bar{\phi}, z)}(f(\bar{\phi}))) \right| \right) = N(\bar{\phi})$$

for some z . Hence

$$\left| \mathcal{E}l(\mathcal{S}_{N(\bar{\phi})} - \text{Sens}_{C(\bar{\phi}, z)}(f(\bar{\phi}))) \right| = 0.$$

Therefore

$$\mathcal{S}_{N(\bar{\phi})} = \text{Sens}_{C(\bar{\phi}, z)}(f(\bar{\phi}))$$

By the above theorem this means $f(\bar{\phi})$ is one-to-one on $C(\bar{\phi}, z)$. So $\bar{\phi}$ is one-to-one on the interior of $\Delta^{n_1-1} \times \Delta^{n_2-1} \times \dots \times \Delta^{n_k-1}$. \square

Due to the above results one can see that it is important to study the minimal elements of the following poset

$$E(\mathcal{S}_n) = \{X \subseteq \mathcal{S}_n \mid \mathcal{E}l(X) = \mathcal{S}_n\}$$

under inclusion.

Example 3.3. When $n = 2$, $\mathcal{S}_2 = \{(0, 1), (1, 0), (1, 1), (1, -1)\}$ and

$$\begin{aligned}\mathcal{E}l(\{(0, 1)\}) &= \{(0, 1), (1, 1), (1, -1)\}, \\ \mathcal{E}l(\{(1, 0)\}) &= \{(1, 0), (1, 1), (1, -1)\}, \\ \mathcal{E}l(\{(1, 1)\}) &= \{(0, 1), (1, 0), (1, 1)\}, \\ \mathcal{E}l(\{(1, -1)\}) &= \{(0, 1), (1, 0), (1, -1)\}.\end{aligned}$$

Therefore $E(\mathcal{S}_n)$ is the set of subsets of \mathcal{S}_2 of size greater than equal to 2 and the minimal elements of $E(\mathcal{S}_2)$ are the subsets of \mathcal{S}_2 of size 2.

Proposition 3.4. *Let $v \in \mathbb{R}^n$ and s be the sign vector of v . If either s or $-s$ is in $\mathcal{E}l(x)$ then $v \cdot x \neq 0$.*

Note that here either s or $-s$ is an element of \mathcal{S}_n .

Proof. Let $s = (s_1, \dots, s_n)$ be the sign vector of $v = (v_1, \dots, v_n)$. Without loss of generality suppose that $s = (s_1, \dots, s_n)$ is in $\mathcal{E}l(x)$ for some $x = (x_1, \dots, x_n) \in \mathcal{S}_n$. Let $\{i_1, \dots, i_j\}$ be the set of all indices for which $s_i \neq 0$ and $x_i \neq 0$. Since $s \in \mathcal{E}l(x)$, there exists $k \in \{\pm 1\}$ such that $s_{i_r} = kx_{i_r}$ for all $1 \leq r \leq j$. By definition $s_i = 0$ if and only if $v_i = 0$. Therefore we have

$$v \cdot x = \sum_{r=1}^j v_{i_r} x_{i_r} = \sum_{r=1}^j |v_{i_r}| s_{i_r} x_{i_r} = k \sum_{r=1}^j |v_{i_r}| (x_{i_r})^2 \neq 0.$$

□

Given a subset $X = \{X^1, \dots, X^m\}$ of \mathcal{S}_n , let M_X be the $(m \times n)$ -matrix whose i -th row is X^i . As an immediate consequence of the above proposition, we have the following results.

Corollary 3.5. *Let $X = \{X^1, \dots, X^m\} \subseteq \mathcal{S}_n$. If the columns of M_X is linearly dependent then $X \notin E(\mathcal{S}_n)$. In particular, if $X \in E(\mathcal{S}_n)$ has size n then X is linearly independent.*

Proof. Let $a_1, \dots, a_n \in \mathbb{R}$ be such that $\sum_{i=1}^n a_i c_i = 0$ where c_i is the i -th column of M_X . Suppose also that the first non-zero term of $a = (a_1, \dots, a_n)$ is positive, that is, $a \in \mathcal{S}_n$. Since $a \cdot X^i = 0$ for $1 \leq i \leq m$, the sign vector s of a is in $\mathcal{S}_n - \mathcal{E}l(X)$ by the above proposition and hence $X \notin E(\mathcal{S}_n)$. □

Corollary 3.6. *If $X \in E(\mathcal{S}_n)$ then $|X| \geq n$.*

Proof. If $|X| < n$ then we can choose a vector v which is orthogonal to all the vectors in X . □

Note that the inequality in the above corollary is strict since $E = \{e_1, \dots, e_n\}$ is in $E(\mathcal{S}_n)$ where $e_k = (0, \dots, 0, \underbrace{1}_k, 0, \dots, 0)$. Since every element of $E(\mathcal{S}_n)$ of size n is minimal by the above corollary, E is indeed a minimal element of $E(\mathcal{S}_n)$.

Let \mathcal{S}_n^0 be the set of all elements of \mathcal{S}_n with non-zero coordinates. These elements are also the ones which eliminates the smallest number of elements of \mathcal{S}_n .

Lemma 3.7. *The subset \mathcal{S}_n^0 is a minimal element of $E(\mathcal{S}_n)$.*

Proof. Since $s = (s_1, \dots, s_n) \in \mathcal{S}_n$ is eliminated by all $t = (t_1, \dots, t_n) \in \mathcal{S}_n^0$ where $t_i = s_i$ whenever $s_i \neq 0$, \mathcal{S}_n^0 is in $E(\mathcal{S}_n)$. On the other hand the element $t \in \mathcal{S}_n^0$ is eliminated by $t' \in \mathcal{S}_n^0$ if and only if $t = t'$. Therefore \mathcal{S}_n^0 is minimal. \square

Example 3.8. Define $T_{\text{red}} = \{0, 1\}$, $T_{\text{green}} = \{0, 1\}$, $T_{\text{blue}} = \{0, 1\}$, and

$$T_{\text{color}} = \left\{ \begin{array}{c} \underbrace{(1, 0, 0, 0)}_{\text{black}}, \underbrace{(0, 1, 0, 0)}_{\text{red}}, \underbrace{(0, 0, 1, 0)}_{\text{green}}, \underbrace{(0, 0, 0, 1)}_{\text{blue}}, \underbrace{(0, 1/2, 0, 1/2)}_{\text{purple}}, \\ \underbrace{(0, 0, 1/2, 1/2)}_{\text{yellow}}, \underbrace{(0, 1/2, 1/2, 0)}_{\text{aqua}}, \underbrace{(0, 1/3, 1/3, 1/3)}_{\text{white}} \end{array} \right\}$$

Let

$$\phi : T_{\text{red}} \times T_{\text{green}} \times T_{\text{blue}} \rightarrow T_{\text{color}}$$

be the multi-valued logic gate which sends the input (r, g, b) to a point which corresponds to the color obtained by mixing the colors whose truth value is 1. For example, $\phi(0, 0, 0) = (1, 0, 0, 0)$, $\phi(0, 1, 0) = (0, 0, 1, 0)$ and $\phi(0, 1, 1) = (0, 0, 1/2, 1/2)$. The induced multi-valued logic gate

$$\bar{\phi} : \Delta^1 \times \Delta^1 \times \Delta^1 \rightarrow \Delta^3$$

is given by

$$\bar{\phi}((r_0, r_1), (g_0, g_1), (b_0, b_1)) = (f_1, f_2, f_3, f_4)$$

where

$$\begin{aligned} f_1 &= r_0 * g_0 * b_0 \\ f_2 &= r_1 * g_0 * b_0 + 1/2 * r_1 * g_0 * b_1 + 1/2 * r_1 * g_1 * b_0 + 1/3 * r_1 * g_1 * b_1 \\ f_3 &= r_0 * g_1 * b_0 + 1/2 * r_0 * g_1 * b_1 + 1/2 * r_1 * g_1 * b_0 + 1/3 * r_1 * g_1 * b_1 \\ f_4 &= r_0 * g_0 * b_1 + 1/2 * r_0 * g_1 * b_1 + 1/2 * r_1 * g_0 * b_1 + 1/3 * r_1 * g_1 * b_1. \end{aligned}$$

Therefore we have

$$\begin{aligned} \frac{\partial}{\partial r_1}(f_2 - f_3 - f_4) &= g_0 * b_0 + g_0 * b_1 + g_1 * b_0 + 2/3 * g_1 * b_1 > 0 \\ \frac{\partial}{\partial g_1}(f_2 - f_3 - f_4) &= -r_0 * b_0 - r_1 * b_0 - 1/3 * r_1 * b_1 < 0 \\ \frac{\partial}{\partial b_1}(f_2 - f_3 - f_4) &= -r_0 * g_0 - r_1 * g_0 - 1/3 * r_1 * g_1 < 0 \end{aligned}$$

and hence

$$\text{Sign}_C(f_2 - f_3 - f_4) = (1, -1, -1)$$

where $C = C(\bar{\phi}, (0, 0, 0))$. Similarly one can show that

$$\begin{aligned} \text{Sign}_C(f_2 + f_3 + f_4) &= (1, 1, 1) \\ \text{Sign}_C(f_2 - f_3 + f_4) &= (1, -1, 1) \\ \text{Sign}_C(f_2 + f_3 - f_4) &= (1, 1, -1). \end{aligned}$$

Since

$$\mathcal{S}_3^0 = \{(1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1)\},$$

we can conclude that $\bar{\phi}$ is a reversible gate by repeated application of the above lemma and the main result of this section.

4. COMPUTING CONTINUOUS SENSITIVITY

We could use experimental data about a multi-valued logic gate to obtain an upper bound on the continuous sensitivity of the gate due to the following simple lemma.

Lemma 4.1. *If X and Z are subsets of \mathcal{S}_N and $Z \cap \mathcal{El}(X) = \emptyset$ then*

$$|\mathcal{El}(Z)| \leq |\mathcal{El}(S_N - \mathcal{El}(X))|$$

In the above lemma consider $X = \text{Sens}_C(f)$ and Z as signs eliminated by experimental data. As an application of this lemma we can show that sensitivity (see Section 2 in [2]) of a boolean function is an upper bound for its continuous sensitivity.

Definition 4.2. Let $\phi : \{0, 1\}^N \rightarrow \{0, 1\}$ be a boolean function and z be an element in $\{0, 1\}^N$. Then the sensitivity of ϕ at the input z is defined as follows:

$$s(\phi, z) = \text{number of indices } i \text{ such that } \phi(z) \neq \phi(z^i)$$

where z^i denotes the element in $\{0, 1\}^N$ obtained by changing the i^{th} coordinate of z . The sensitivity of ϕ is defined as follows:

$$s(\phi) = \max \{ s(\phi, z) \mid z \in \{0, 1\}^N \}.$$

Theorem 4.3. *Let $\phi : \{0, 1\}^N \rightarrow \{0, 1\}$ be a boolean function. Then*

$$cs(\bar{\phi}) \leq s(\phi).$$

Proof. Let $z = (z_1, z_2, \dots, z_N)$ be an element in $\{0, 1\}^N$. Define $D_z = \{i \mid \phi(z) = \phi(z^i)\}$. Then we have

$$s(\phi, z) = N - |D_z|.$$

Notice that by Mean Value Theorem for every i in D_z there exists $c_{i,z}$ in $C(\bar{\phi}, z)$ such that

$$\frac{\partial \bar{\phi}}{\partial x_i}(z_1, z_2, \dots, z_{i-1}, c_{i,z}, z_{i+1}, z_{i+2}, \dots, z_N) = 0,$$

hence e_i is not $\text{Sens}_{C(\bar{\phi}, z)}(f(\bar{\phi}))$ for all i in D_z . Therefore, we have

$$\left| \mathcal{E}l(\mathcal{S}_N - \text{Sens}_{C(\bar{\phi}, z)}(f(\bar{\phi}))) \right| \geq |\mathcal{E}l(\{e_i \mid i \in D_z\})| = \frac{3^N - 3^{N-|D_z|}}{2}$$

and hence

$$cs(\bar{\phi}, z) = \log_3 \left(3^N - 2 \left| \mathcal{E}l(\mathcal{S}_N - \text{Sens}_{C(\bar{\phi}, z)}(f(\bar{\phi}))) \right| \right) \leq N - |D_z| = s(\phi, z)$$

Therefore the result follows. \square

The above result shows that it is important to know how to count eliminated signs. For the rest of this section we will discuss methods for counting eliminated signs. Notice that the number of elements eliminated by each element of \mathcal{S}_2 is the same. This is not true in general. For example for $x = (0, 0, 1)$ and $y = (1, 1, 1)$ in \mathcal{S}_3 , we have

$$\begin{aligned} \mathcal{E}l(\{x\}) &= \mathcal{S}_3 - \{(0, 1, 0), (1, 0, 0), (1, 1, 0), (1, -1, 0)\}, \\ \mathcal{E}l(\{y\}) &= \mathcal{S}_3 - \{(0, 1, -1), (1, 0, -1), (1, 1, -1), (1, -1, 0), (1, -1, 1), (1, -1, -1)\} \end{aligned}$$

and hence $|\mathcal{E}l(\{x\})| \neq |\mathcal{E}l(\{y\})|$. In general the size of $\mathcal{E}l(\{x\})$ depends only on the number of zeros of x and is given as follows.

Lemma 4.4. *For $x \in \mathcal{S}_n$, $|\mathcal{E}l(\{x\})| = 3^{z(x)}(2^{n-z(x)} - 1)$ where $z(x)$ is the number of zeros of x .*

Proof. Let $x = (x_1, \dots, x_n) \in \mathcal{S}_n$. We first consider the case $z(x) = 0$. In this case x eliminates $s = (1, s_2, \dots, s_n)$ if and only if $s_i = 0$ or x_i for $i \geq 2$. If the first nonzero term of s is k -th one then x eliminates s either $s_i \in \{0, x_i\}$ or $s_i \in \{0, -x_i\}$. Therefore when $z(x) = 0$, we have

$$|\mathcal{E}l(\{x\})| = 2^{n-1} + \sum_{k=2}^n 2^{n-k} = 2^n - 1.$$

Now suppose that $z(x) \neq 0$. We prove this case by induction on n . The case $n = 2$ follows from Example 3.3. Let $x' = (x_1, \dots, \hat{x}_j, \dots, x_n) \in \mathcal{S}_{n-1}$ where $j = \min\{i \mid t_i = 0\}$. When $j = 1$

$$\mathcal{E}l(\{x\}) = \{(\varepsilon, s_1, \dots, s_{n-1}), (1, -s_1, \dots, -s_{n-1}) \mid \varepsilon \in \{0, 1\}, (s_1, \dots, s_{n-1}) \in \mathcal{E}l(x')\},$$

and otherwise we have

$$\mathcal{E}l(\{x\}) = \{(s_1, \dots, \varepsilon, \dots, s_{n-1}) \mid \varepsilon \in \{-1, 0, 1\}, (s_1, \dots, s_{n-1}) \in \mathcal{E}l(x')\}.$$

Therefore the number of elements of $\mathcal{E}l(x)$ is three times the number of elements of $\mathcal{E}l(x')$ and hence the result follows by induction. \square

Now we generalize the above lemma to the intersections of eliminated sets for m -many elements in \mathcal{S}_n . Let z_X be the number of zero columns of M_X .

Theorem 4.5. For $X = \{X^1, \dots, X^m\} \subseteq \mathcal{S}_n$, we have

$$\left| \bigcap_{i=1}^m \mathcal{E}l(X^i) \right| = 3^{zx} \left(-(-2)^{m-1} + \sum_{\alpha \in \mathcal{S}_m} (-1)^{z(\alpha)} 2^{z(\alpha) + |\text{BSp}(X, \alpha)|} \right)$$

where for any $\alpha \in \mathcal{S}_m$, $\text{BSp}(X, \alpha)$ is the set of columns of M_X of the form $\pm \sum_{i=1}^m a_i \alpha_i e_i$ with $a_i \in \{0, 1\}^m - \{(0, \dots, 0)\}$.

We prove this theorem using the following lemma.

Lemma 4.6. Let $X = \{X^1, \dots, X^m\} \subseteq \mathcal{S}_n$ where $X^j = (X_1^j, \dots, X_n^j)$ be such that M_X has no zero column. Then the number of elements of the form $(1, s_2, \dots, s_n)$ in $\bigcap_{i=1}^m \mathcal{E}l(X^i)$ is

$$\sum_{\alpha \in A} (-1)^{z(\alpha)} 2^{z(\alpha) + |\text{BSp}(X, \alpha)| - 1}$$

where

$$A = \{(\alpha_1, \dots, \alpha_m) \in \mathcal{S}_m \mid \alpha_j = 1 \text{ whenever } X_1^j \neq 0\}.$$

Proof. Let $c^i = (X_1^i, \dots, X_n^i)^T$ be the i -th column of M_X for $1 \leq i \leq n$. By reordering elements of X , we can assume that $c^1 = e_1^T + \dots + e_r^T$ for some $1 \leq r \leq m$. In this case, we have $A = \{(1, \dots, 1, \alpha_{r+1}, \dots, \alpha_m) \mid \alpha_{r+i} \in \{-1, 0, 1\}\}$. Note that the i -th column c^i is not in $\bigcup_{\alpha \in A} \text{Bsp}(X, \alpha)$ if and only if $c_j^i = 1$ and $c_{j'}^i = -1$ for some

$1 \leq j, j' \leq r$. Moreover if $s = (1, s_2, \dots, s_n) \in \bigcap_{i=1}^m \mathcal{E}l(X^i)$ then $s_i = 0$ for all i for which $c^i \notin \bigcup_{\alpha \in A} \text{Bsp}(X, \alpha)$. So without loss of generality we can assume that $c^i \in \text{Bsp}(X, \alpha)$

for some $\alpha = (1, \dots, 1, \alpha_{r+1}, \dots, \alpha_m) \in \mathcal{S}_m$ for all i , that is, $c^i = \pm \sum a_j^i \alpha_j e_j$ where $a_j^i \in \{0, 1\}$ for $1 \leq i \leq m$.

For each $\alpha = (1, \dots, 1, \alpha_{r+1}, \dots, \alpha_m) \in \mathcal{S}_m$ with $\alpha_i \neq 0$ for any i , let A_α be the set of all $s = (1, s_2, \dots, s_n)$ where $s_i \in \{0, 1\}$ if $c^i = \sum a_j^i \alpha_j e_j$ and $s_i \in \{0, -1\}$ if $c^i = -\sum a_j^i \alpha_j e_j$. Here, $|A_\alpha| = 2^{|\text{BSp}(X', \alpha)|}$. An element $s = (1, s_2, \dots, s_n) \in \mathcal{S}_n$ lies in the intersection of A_α and $\bigcap_{i=1}^m \mathcal{E}l(X^i)$ if and only if s and X^i has common non-zero elements for each $r+1 \leq i \leq m$. Clearly, $s \in A_\alpha$ does not satisfy this property if there exists $r+1 \leq k \leq m$ such that $a_k^i \neq 0$ implies $s^i = 0$. To eliminate these terms, we first need to remove the ones with $s_i = 0$ for all $a_k^i \neq 0$ for all $r+1 \leq k \leq m$. For each k , there are $2^{|\text{BSp}(X', \alpha^k)|}$ many such s in A_α where $\alpha_j^k = \alpha_j$ if $j \neq k$ and $\alpha_k^k = 0$. Then we need to add the ones with $s_i = 0$ for all $a_k^i \neq 0$ or $a_{k'}^i \neq 0$ for $r+1 \leq k, k' \leq m$ since we remove them twice. There are $2^{|\text{BSp}(X', \alpha^{k, k'})|}$ many such s in A_α where $\alpha_j^{k, k'} = \alpha_j$ if $j \neq k$ or k' and $\alpha_k^{k, k'} = \alpha_{k'}^{k, k'} = 0$. Then we need to remove the ones corresponding to triples since we add them twice. By continuing in this way,

we obtain that

$$|A_\alpha \cap \left(\bigcap_{i=1}^m \mathcal{E}l(X^i) \right)| = \sum_{\beta \in S_\alpha} (-1)^{z(\beta)} 2^{|\text{BSp}(X', \beta)|}$$

where $S_\alpha = \{\beta = (1, \dots, 1, \beta_{r+1}, \dots, \beta_m) \mid \beta_j = \alpha_j \text{ or } 0, r+1 \leq j \leq m\}$.

Let $\alpha = (1, \dots, 1, \alpha_{r+1}, \dots, \alpha_m)$ and $\gamma = (1, \dots, 1, \gamma_{r+1}, \dots, \gamma_m)$ be distinct elements of $\{\pm 1\}^m$, i.e, there exist k such that $\alpha_k = -\gamma_k$. If $s \in A_\alpha \cap A_\gamma$ then whenever $a_k^i \neq 0$. So we have $A_\alpha \cap A_\gamma \cap \left(\bigcap_{i=1}^m \mathcal{E}l(X^i) \right) = \emptyset$. Clearly, if $s = (1, s_2, \dots, s_n)$ is in the intersection of $\mathcal{E}l(X^i)$'s then $s \in A_\alpha$. Therefore we have

$$\left| \bigcap_{i=1}^m \mathcal{E}l(X^i) \right| = \sum_{\alpha = (1, \dots, 1, \alpha_{r+1}, \dots, \alpha_m) \in \{\pm 1\}^m} \sum_{\beta \in S_\alpha} (-1)^{z(\beta)} 2^{|\text{BSp}(X', \beta)|}.$$

Since β is an element of S_α for $2^{z(\beta)}$ -many distinct α 's, we have

$$\left| \bigcap_{i=1}^m \mathcal{E}l(X^i) \right| = \sum_{\beta = (1, \dots, 1, \beta_{r+1}, \dots, \beta_m) \in S_m} (-1)^{z(\beta)} 2^{z(\beta) + |\text{BSp}(X', \beta)|}.$$

Since $|\text{BSp}(X', \beta)| = |\text{BSp}(X, \beta)| - 1$, the result follows. \square

Proof of Theorem 4.5. Let $X^i = (X_1^i, \dots, X_n^i)$ for $1 \leq i \leq m$. We proceed by induction on n . The case $n = 2$ follows from Example 3.3. Note that in this case the size of the intersection of eliminated set of two different elements of \mathcal{S}_2 is 2, three different elements is 1 and the intersection of eliminated sets of all is 0. For $n > 2$, we first consider the case where M_X has a zero column. Let $\tilde{X} = \{\tilde{X}^1, \dots, \tilde{X}^m\}$ where $\tilde{X}^i = (X_1^i, \dots, X_{k-1}^i, X_{k+1}^i, \dots, X_n^i)$ if k -the column of M_X is zero. Then

$$\left| \bigcap_{i=1}^m \mathcal{E}l(\tilde{X}^i) \right| = 3^{z_{\tilde{X}}} \left(-(-2)^{m-1} + \sum_{\alpha \in \mathcal{S}_m} (-1)^{z(\alpha)} 2^{z(\alpha) + |\text{BSp}(\tilde{X}, \alpha)|} \right)$$

by induction hypothesis. Note that if $k \neq 1$ then $s = (s_1, \dots, s_n)$ is in $\bigcap_{i=1}^m \mathcal{E}l(X)$ if

and only if $(s_1, \dots, s_{k-1}, s_{k+1}, \dots, s_n) \in \bigcap_{i=1}^m \mathcal{E}l(\tilde{X}^i)$, and $s_k \in \{0, 1, -1\}$. If $k = 1$

then the elements in $\bigcap_{i=1}^m \mathcal{E}l(X)$ are of the form $(\varepsilon, s_2, \dots, s_n)$ where $\varepsilon \in \{0, 1\}$ and

$(s_2, \dots, s_n) \in \bigcap_{i=1}^m \mathcal{E}l(\tilde{X})$ or $\varepsilon = -1$ and $-(s_2, \dots, s_n) \in \bigcap_{i=1}^m \mathcal{E}l(\tilde{X})$. So the result

follows for this case since $\text{BSp}(X, \alpha) = \text{BSp}(\tilde{X}, \alpha)$.

Now, suppose that M_X has no zero column. By reordering elements of X , we can assume that first column of M_X is of the form $e_1 + \dots + e_r$ for some $1 \leq r \leq m$. Let i_1, \dots, i_k be the set of all i 's for which $(X_2^i, \dots, X_n^i) \notin \mathcal{S}_{n-1}$. Clearly, $1 \leq i_1, \dots, i_k \leq$

r . Then $\overline{X} = \{\overline{X^1}, \dots, \overline{X^m}\} \subset \mathcal{S}_{n-1}$ where $\overline{X^j} = (X_2^j, \dots, X_n^j)$ if $j \neq i_t$ for some $1 \leq t \leq k$ and $\overline{X^j} = -(X_2^j, \dots, X_n^j)$, otherwise. Then $s = (0, s_2, \dots, s_n) \in \mathcal{S}_n$ is in $\bigcap_{i=1}^m \mathcal{E}l(X^i)$ if and only if (s_2, \dots, s_{n-1}) is in $\bigcap_{i=1}^m \overline{X^i}$. Therefore the size of the elements of this form in $\bigcap_{i=1}^m \mathcal{E}l(X^i)$ is

$$-(-2)^{m-1} + \sum_{\alpha \in \mathcal{S}_m} (-1)^{z(\alpha)} 2^{z(\alpha) + |\text{BSp}(\overline{X}, \alpha)|}$$

by induction. Note that for $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\alpha' = (\alpha'_1, \dots, \alpha'_m)$ in \mathcal{S}_m with $\alpha'_j = -\alpha_j$ for $j = i_1, \dots, i_k$ and $\alpha'_j = \alpha_j$ otherwise, we have

$$|\text{BSp}(\overline{X}, \alpha)| = |\text{BSp}(X, \alpha')| \text{ and } z(\alpha) = z(\alpha').$$

Moreover for $\alpha = (1, \dots, 1, \alpha_{r+1}, \dots, \alpha_m) \in \mathcal{S}_m$, $|\text{BSp}(\overline{X}, \alpha)| = |\text{BSp}(X, \alpha)| - 1$.

On the other hand, there are

$$\sum_{\alpha = (1, \dots, 1, \alpha_{r+1}, \dots, \alpha_m) \in \mathcal{S}_m} (-1)^{z(\alpha)} 2^{z(\alpha) + |\text{BSp}(X, \alpha)| - 1}$$

many elements of the form $s = (1, s_2, \dots, s_n) \in \mathcal{S}_n$ is in $\bigcap_{i=1}^m \mathcal{E}l(X^i)$ by previous lemma. Therefore the total number of elements in the intersection of the $\mathcal{E}l(X^i)$'s is

$$-(-2)^{m-1} + \sum_{\alpha \in \mathcal{S}_m} (-1)^{z(\alpha)} 2^{z(\alpha) + |\text{BSp}(X, \alpha)|}$$

in this case. □

Now one can use the inclusion-exclusion principle, to find the number of elements eliminated by an arbitrary subset $X = \{X^1, \dots, X^t\}$ of \mathcal{S}_n .

Proposition 4.7. *The size of the set of elements eliminated by $X = \{X^1, \dots, X^m\}$ is*

$$\sum_{k=1}^m \sum_{1 \leq i_1 < \dots < i_k \leq m} 3^{z_{X(i_1, \dots, i_k)}} \left(-2^{k-1} + \sum_{\alpha \in \mathcal{S}_k} (-1)^{z(\alpha) + k + 1} 2^{z(\alpha) + |\text{BSp}(X(i_1, \dots, i_k), \alpha)|} \right)$$

where $X(i_1, \dots, i_k) = \{X^{i_1}, \dots, X^{i_k}\}$.

To make the calculations easier, we introduce two column operations on the set of matrices of the form M_X . First one is the action of the symmetric group S_n of order n . We define the action of $\sigma \in S_n$ on \mathcal{S}_n by

$$\sigma(x) = \begin{cases} (x_{\sigma(1)}, \dots, x_{\sigma(n)}), & \text{if } (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \in \mathcal{S}_n; \\ (-x_{\sigma(1)}, \dots, -x_{\sigma(n)}), & \text{otherwise.} \end{cases}$$

Here $(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \in \mathcal{S}_n$ means that the first non-zero term of $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ is 1. This induces an action of σ on $\{M_X \mid X \subseteq \mathcal{S}_n\}$ by sending M_X to $M_{\sigma(X)}$ where

$\sigma(X) = \{\sigma(X^1), \dots, \sigma(X^n)\}$. The second operation is multiplying a column of M_X by -1 . We also need to be careful here. For $x = (x_1, \dots, x_n) \in \mathcal{S}_n$, let

$$x_j^- = \begin{cases} (x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n), & \text{if } (x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n) \in \mathcal{S}_n; \\ -(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n), & \text{otherwise.} \end{cases}$$

Here what we mean by the matrix obtained by multiplying j -th column of M_X by -1 is the matrix $M_{X[j]}$ where $X[j] = \{(X^1)_j^-, \dots, (X^n)_j^-\}$. Clearly these operations preserves the cardinality of arbitrary intersections of eliminated sets.

Example 4.8. Let $X = \{x, y\} \subset \mathcal{S}_n$. By applying column operations, we can assume that M_X has 5 types of columns namely

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Suppose that M_X has a_1 -many columns of first type, a_2 -many columns of second type, b_1 -many columns of third type, b_2 -many columns of forth type and c -many columns of last type. This means that x has $(b_2 + c)$ -many zeros and y has $(b_1 + c)$ -many zeros. Therefore

$$|\mathcal{E}l(x) \cap \mathcal{E}l(y)| = 3^c \left(2^{b_1+b_2} (2^{a_1} + 2^{a_2}) - 2^{b_1+1} - 2^{b_2+1} + 2 \right)$$

and hence

$$\begin{aligned} |\mathcal{E}l(X)| &= |\mathcal{E}l(x)| + |\mathcal{E}l(y)| - |\mathcal{E}l(x) \cap \mathcal{E}l(y)| \\ &= 3^c \left(3^{b_1} 2^{a_1+a_2+b_2} + 3^{b_2} 2^{a_1+a_2+b_1} - 2^{b_1+b_2} (2^{a_1} + 2^{a_2}) - 3^{b_1} - 3^{b_2} + 2^{b_1+1} + 2^{b_2+1} - 2 \right) \end{aligned}$$

REFERENCES

- [1] Anas N. Al-Rabadi. *Reversible logic synthesis*. Springer-Verlag, Berlin, 2004. From fundamentals to quantum computing.
- [2] Andris Ambainis, Mohammad Bavarian, Yihan Gao, Jieming Mao, Xiaoming Sun, and Song Zuo. Tighter relations between sensitivity and other complexity measures. In *Automata, languages, and programming. Part I*, volume 8572 of *Lecture Notes in Comput. Sci.*, pages 101–113. Springer, Heidelberg, 2014.
- [3] F. Bacsó. Derivation of vector-valued Boolean functions. *Acta Math. Hungar.*, 87(3):197–203, 2000.
- [4] C. H. Bennett. Logical reversibility of computation. *IBM J. Res. Develop.*, 17:525–532, 1973.
- [5] Leonard Bolc and Piotr Borowik. *Many-valued logics. Vol. 1*. Springer-Verlag, Berlin, 1992. Theoretical foundations.
- [6] J. Bourgain. On the distributions of the Fourier spectrum of Boolean functions. *Israel J. Math.*, 131:269–276, 2002.
- [7] Jehoshua Bruck. Harmonic analysis of polynomial threshold functions. *SIAM J. Discrete Math.*, 3(2):168–177, 1990.
- [8] Daizhan Cheng and Xiangru Xu. Bi-decomposition of multi-valued logical functions and its applications. *Automatica J. IFAC*, 49(7):1979–1985, 2013.

- [9] Yves Crama and Peter L. Hammer. *Boolean functions*, volume 142 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2011. Theory, algorithms, and applications.
- [10] Craig Gotsman and Nathan Linial. Spectral properties of threshold functions. *Combinatorica*, 14(1):35–50, 1994.
- [11] Ana Graça, João Marques-Silva, Inês Lynce, and Arlindo L. Oliveira. Haplotype inference with pseudo-Boolean optimization. *Ann. Oper. Res.*, 184:137–162, 2011.
- [12] Claire Kenyon and Samuel Kutin. Sensitivity, block sensitivity, and l -block sensitivity of Boolean functions. *Inform. and Comput.*, 189(1):43–53, 2004.
- [13] Martin Kutrib. Aspects of reversibility for classical automata. In *Computing with new resources*, volume 8808 of *Lecture Notes in Comput. Sci.*, pages 83–98. Springer, Cham, 2014.
- [14] R. Landauer. Irreversibility and heat generation in the computing process. *IBM J. Res. Develop.*, 5:183–191, 1961.
- [15] V. K. Leont’ev. On pseudo-Boolean polynomials. *Comput. Math. Math. Phys.*, 55(11):1926–1932, 2015.
- [16] Noam Nisan. CREW PRAMs and decision trees. *SIAM J. Comput.*, 20(6):999–1007, 1991.
- [17] Noam Nisan and Márió Szegedy. On the degree of Boolean functions as real polynomials. *Comput. Complexity*, 4(4):301–313, 1994. Special issue on circuit complexity (Barbados, 1992).
- [18] Ryan O’Donnell. Some topics in analysis of Boolean functions. In *STOC’08*, pages 569–578. ACM, New York, 2008.
- [19] Giovanni Panti. Multi-valued logics. In *Quantified representation of uncertainty and imprecision*, volume 1 of *Handb. Defeasible Reason. Uncertain. Manag. Syst.*, pages 25–74. Kluwer Acad. Publ., Dordrecht, 1998.
- [20] Ian Parberry. *Circuit complexity and neural networks*. Foundations of Computing Series. MIT Press, Cambridge, MA, 1994.
- [21] Asher Peres. Reversible logic and quantum computers. *Phys. Rev. A (3)*, 32(6):3266–3276, 1985.
- [22] Emil L. Post. Introduction to a General Theory of Elementary Propositions. *Amer. J. Math.*, 43(3):163–185, 1921.
- [23] Barkley Rosser. On the many-valued logics. *Amer. J. Phys.*, 9:207–212, 1941.
- [24] J. Barkley Rosser and Atwell R. Turquette. *Many-valued logics*. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1951.
- [25] Sergiu Rudeanu. Local properties of Boolean functions. I. Injectivity. *Discrete Math.*, 13(2):143–160, 1975.
- [26] Sergiu Rudeanu. Local properties of Boolean functions. II. Isotony. *Discrete Math.*, 13(2):161–183, 1975.
- [27] Karl Schmidt. The theory of functions of one Boolean variable. *Trans. Amer. Math. Soc.*, 23(2):212–222, 1922.
- [28] I. M. Sobol’ and S. Kucherenko. Derivative based global sensitivity measures and their link with global sensitivity indices. *Math. Comput. Simulation*, 79(10):3009–3017, 2009.
- [29] I. M. Sobol’ and S. S. Kucherenko. On global sensitivity analysis of quasi-Monte Carlo algorithms. *Monte Carlo Methods Appl.*, 11(1):83–92, 2005.
- [30] Tommaso Toffoli. Computation and construction universality of reversible cellular automata. *J. Comput. System Sci.*, 15(2):213–231, 1977.
- [31] Tommaso Toffoli. Reversible computing. In *Automata, languages and programming (Proc. Seventh Internat. Colloq., Noordwijkerhout, 1980)*, volume 85 of *Lecture Notes in Comput. Sci.*, pages 632–644. Springer, Berlin-New York, 1980.

- [32] John von Neumann. The general and logical theory of automata. In *Cerebral Mechanisms in Behavior. The Hixon Symposium*, pages 1–31; discussion, pp. 32–41. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951.
- [33] Pan Wang, Zhenzhou Lu, Jixiang Hu, and Changcong Zhou. Sensitivity analysis of the variance contributions with respect to the distribution parameters by the kernel function. *Comput. Math. Appl.*, 67(10):1756–1771, 2014.
- [34] Satoshi Watanabe. Reversibility of quantum electrodynamics. *Physical Rev. (2)*, 84:1008–1025, 1951.
- [35] Ingo Wegener. The critical complexity of all (monotone) Boolean functions and monotone graph properties. *Inform. and Control*, 67(1-3):212–222, 1985.
- [36] Shigeru Yamashita and Shin-ichi Minato, editors. *Reversible computation*, volume 8507 of *Lecture Notes in Computer Science*. Springer, Cham, 2014.

ASLI GÜÇLÜKAN İLHAN, DEPARTMENT OF MATHEMATICS, DOKUZ EYLÜL UNIVERSITY, BUCA, İZMİR, TURKEY.

E-mail address: `asli.ilhan@deu.edu.tr`

ÖZGÜN ÜNLÜ, DEPARTMENT OF MATHEMATICS, BILKENT UNIVERSITY, BILKENT, ANKARA, TURKEY.

E-mail address: `unluo@fen.bilkent.edu.tr`